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Anomalous scaling of a passive vector advected by the Navier–Stokes velocity field

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Abstract

Using the field theoretic renormalization group and the operator-product expansion, the model of a passive vector field (a weak magnetic field in the framework of the kinematic MHD) advected by the velocity field which is governed by the stochastic Navier–Stokes equation with the Gaussian random stirring force δ-correlated in time and with the correlator proportional to $k^{4-d-2\varepsilon}$ is investigated to the first order in ε (one-loop approximation). It is shown that the single-time correlation functions of the advected vector field have anomalous scaling behavior and the corresponding exponents are calculated in the isotropic case, as well as in the case with the presence of large-scale anisotropy. The hierarchy of the anisotropic critical dimensions is briefly discussed and the persistence of the anisotropy inside the inertial range is demonstrated on the behavior of the skewness and hyperskewness (dimensionless ratios of correlation functions) as functions of the Reynolds number Re. It is shown that even though the present model of a passive vector field advected by the realistic velocity field is mathematically more complicated than, on one hand, the corresponding models of a passive vector field advected by 'synthetic' Gaussian velocity fields and, on the other hand, than the corresponding model of a passive scalar quantity advected by the velocity field driven by the stochastic Navier–Stokes equation, the final one-loop approximate asymptotic scaling behavior of the single-time correlation or structure functions of the advected fields of all models are defined by the same anomalous dimensions (up to normalization).

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1. Introduction

Theoretical explanation of possible deviations from the classical phenomenological Kolmogorov–Obukhov (KO) theory which is suggested by both natural and numerical experiments [1–5] still remains one of the most interesting open problems in fully developed turbulence and related models.

According to the first and second Kolmogorov hypotheses of the KO theory, the single-time structure functions of the velocity field in the inertial range ($l \ll r \ll L$)

$$S_N(r) = \langle [v_r(t, \mathbf{x}) - v_r(t, \mathbf{x}')]^N \rangle, \qquad r = |\mathbf{x} - \mathbf{x}'| \tag{1}$$

are independent of both the external (integral) scale L and internal (viscous) scale l, where v_r denotes the component of the velocity field directed along the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. Simple dimensional analysis then leads to the scale-invariant form of the structure functions (1)

$$S_N(r) = \text{const} \times (\bar{\epsilon} r)^{N/3}, \tag{2}$$

where $\bar{\epsilon}$ is the mean dissipation rate.

But, as was mentioned above, both experimental and theoretical investigations lead to deviations from the KO theory and, in contradiction with the first Kolmogorov hypothesis [1, 3, 6], the inertial-range behavior of the structure functions (1) must be modified as follows:

$$S_N(r) = (\bar{\epsilon} r)^{N/3} R_N(r/L), \tag{3}$$

with some scaling functions R_N. It is supposed that they have powerlike asymptotic behavior in the region $r/L \ll 1$, namely,

$$R_N(r/L) \simeq \text{const} \times (r/L)^{q_N}. \tag{4}$$

The singular dependence of the structure functions on L in the limit $L \to \infty$ together with nonlinearity of the exponents q_N as functions of N is called 'anomalous scaling'. Such a kind of behavior is theoretically explained by strongly developed fluctuations of the dissipation rate which is encoded in the concept intermittency [1–5].

The effective method for the investigation of a self-similar scaling behavior is the renormalization group (RG) [7, 8]. If one applies it to the fully developed turbulence based on the stochastic Navier–Stokes equation [9–12] then the second Kolmogorov hypothesis can be proved for a variety realistic random forces [12] with the following infrared (IR) ($r \gg l$) scaling critical dimensions for the structure functions (1)

$$\Delta[S_N] = -N/3, \tag{5}$$

which are given as coefficients of the corresponding RG equations and are in agreement with the simple dimensional analysis (2). On the other hand, to analyze the behavior of the scaling functions $R_N(r/L)$ in the limit $r/L \to 0$ it is necessary to go beyond the standard RG analysis and to use the operator-product expansion (OPE) [7, 8, 11, 12]. Application of the OPE leads to the following powerlike representation of the scaling functions (4):

$$R_N(r/L) = \sum_F C_F (r/L)^{\Delta_F}, \qquad r/L \to 0, \tag{6}$$

where the summation is implied over some class of composite operators F, Δ_F are their critical dimensions, and C_F are coefficients regular in L^{-1}. The anomalous scaling in the theory of fully developed turbulence based on the stochastic Navier–Stokes equation is related to the existence of the composite operators with the negative critical dimensions. These so-called 'dangerous operators' lead to singular behavior of the structure functions in the limit $r/L \to 0$ [12] and, roughly speaking, the final asymptotic anomalous behavior is determined by the smallest (the most negative) critical dimension. But in the stochastic Navier–Stokes model dangerous operators enter into the OPE in the form of the infinite families with the spectrum of critical dimensions unbounded from below. It means that the nontrivial problem of the summation of their contributions arises. This is still an unsolved problem of the theory.

Nevertheless, during the last two decades great progress was achieved in the theoretical understanding of the anomalous scaling of the single-time structure or correlation functions of passive scalar or vector fields advected by given Gaussian statistics of the velocity field. Although such kind of models is much more easier from a theoretical point of view, the deviations from the KO theory are surprisingly more noticeable and visible for them than for the velocity field itself (see, e.g., [13–18]). The main role in these investigations was played by the so-called 'rapid change model' of a passively advected scalar field by a self-similar Gaussian δ-correlated in time velocity field introduced by Kraichnan [19]. Namely, in this model, for the first time, the systematic analysis of the corresponding anomalous exponents was done on the microscopic level. For example, within the so-called 'zero-mode approach' to the rapid change model [20] (see also survey paper [5]) the anomalous exponents are found from the homogeneous solutions (zero modes) of the closed equations for the single-time correlations. The corresponding analysis of the rapid-change model for the vector (magnetic) field which was introduced in [21] can be found in [22–24].

The model was also intensively investigated by the field theoretic RG technique, where systematic perturbation expansion for the anomalous exponents was constructed, and the exponents were calculated to the second [25] and third [26] orders. Besides, various descendants of the Kraichnan rapid-change model, namely, models with inclusion of small-scale anisotropy [27], compressibility [28], the finite correlation time of velocity field [29, 30] and helicity [31] were studied by the field theoretic RG approach. Moreover, advection of the passive vector field by the Gaussian self-similar velocity field (with and without large- and small-scale anisotropies, pressure, compressibility and a finite correlation time) has been also investigated, all possible asymptotic scaling regimes and crossover among them have been classified, and anomalous scaling was analyzed [24, 32, 33] (see also survey paper [34]). The main conclusion of all these studies is that the anomalous scaling remains valid for all generalized models.

However, although the advection models with the so-called 'synthetic' velocity field describe nicely many of the anomalous features of the genuine turbulent advection of the scalar or vector quantities, they still remain artificial models, and they can be considered only as the first step of the investigation of intermittency and anomalous scaling of a scalar or vector field advection in fully developed turbulence. Moreover, they have some important drawbacks. For example, the models with the finite correlation time of the velocity field are Galilean non-invariant [15]. As a consequence, they do not take into account the self-advection of turbulent eddies. As a result of these so-called 'sweeping effects' the different time correlations of the Eulerian velocity are not self-similar and depend strongly on the integral scale [35]. Thus, as was discussed in [29], the perturbative expansion in the parameter which characterizes the energy spectrum of the velocity field in the inertial range is potentially dangerous even in the case with Gaussian spatial statistics of the velocity field. On the other hand, the Kraichnan rapid-change model is Galilean invariant and is free of sweeping effects but the model is so simple that it is not possible to describe some features of genuine turbulence within it (e.g., helical effects cannot be investigated within the Kraichnan model [31]).

Therefore, the crucial point for further progress in understanding the anomalous scaling at the microscopic level is to go beyond Gaussianity of the velocity field and investigate the anomalous scaling of the single-time structure or correlation functions of scalar or vector fields advected by a non-Gaussian velocity field governed by the stochastic Navier–Stokes equation.

In the case of a scalar-field advection the first steps in this direction were already done in [36], where the anomalous exponents for the single-time structure functions of the scalar field were calculated to the second order in the corresponding perturbation theory. It was shown that the mechanism of the origin of the anomalous scaling is the same as in the case of the Kraichnan model with Gaussian statistics of the velocity field, namely, it is related to the existence in the model of composite operators with negative scaling dimensions. Further, it was shown that in the first-order perturbation theory the anomalous exponents are the same as in the case of the Kraichnan rapid-change model (up to a normalization) but in the second order they are sufficiently different. On the other hand, the systematic analytical analysis of the anomalous scaling of correlation functions of a passive vector field advected by the Navier–Stokes velocity field absents at all, although conclusions of such investigations can be interesting from experimental, as well as theoretical point of view. In this respect, maybe the most interesting question, which still waits for an answer, sounds as follows. Is there nontrivial dependence of anomalous exponents of the corresponding correlation functions on the internal (tensor) structure of the advected field?

In this paper the first step will be to answer the above question, namely, we shall analyze the spatial structure of the single-time correlation functions of a passive vector quantity (for example, weak magnetic field) advected by the incompressible velocity field driven by the stochastic Navier–Stokes equation with a given random stirring force in the presence of large-scale anisotropy. The model is also known as the kinematic MHD turbulence because it can be obtained from MHD turbulence when one omits the Lorentz force term in the equation for the velocity field. This is also the reason why, starting from the next section, we are working in terms of the kinematic MHD turbulence. Our aim is to calculate the corresponding anomalous exponents in the first order in ε and to compare the results, on one hand, with the anomalous exponents obtained within the models with Gaussian statistics of the velocity field [24, 32, 33] and, on the other hand, with the anomalous exponents obtained in the model of passive scalar advected by the velocity field driven by the stochastic Navier–Stokes equation [36].

The main result of this paper is the conclusion that, despite the fact that the present model of a passively advected vector field by the velocity field driven by the stochastic Navier–Stokes equation is more complicated from a mathematical point of view than the corresponding model of a passive scalar advection, the anomalous asymptotic behavior of the single-time correlation functions (in the case of an advected vector field) and of the single-time structure functions (in the case of an advected scalar field) is given by the same anomalous dimensions of the corresponding composite operators in the isotropic case, as well as in the case with the presence of large-scale anisotropy, at least, within the one-loop approximation.

The model is not interesting only from purely theoretical point of view but it has also possible practical application in cosmic physics when one can meet situations when relatively weak small-scale magnetic field evolves on the background of a strong large-scale uniaxial magnetic field (see, e.g., [37] and also discussion in [24]). One of the interesting questions which arises in such situations is the question about the influence of the large-scale anisotropy generated by the large-scale magnetic field on the small-scale statistics of the advected magnetic field. As was already discussed above, this question is also the main reason for this paper.

This paper is organized as follows. In section 2, the model is defined and the field theoretic formulation of the model is given. In section 3, we perform the ultraviolet (UV) renormalization of the model and the stability of the scaling regime is discussed. In section 4, the explicit from of the anomalous exponents is found. Obtained results are reviewed and discussed in section 5.

2. Field-theoretic formulation of the model of kinematic MHD turbulence

The advection of a passive solenoidal magnetic field $\mathbf{b} \equiv \mathbf{b}(x)$ $(x \equiv (t, \mathbf{x}))$ in the framework of the kinematic MHD model is described by the following system of stochastic equations

$$\partial_t \mathbf{b} = \nu_0 u_0 \triangle \mathbf{b} - (\mathbf{v} \cdot \boldsymbol{\partial})\mathbf{b} + (\mathbf{b} \cdot \boldsymbol{\partial})\mathbf{v} + \mathbf{f}^{\mathbf{b}}, \tag{7}$$

$$\partial_t \mathbf{v} = \nu_0 \triangle \mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\partial})\mathbf{v} - \partial P + \mathbf{f}^{\mathbf{v}}, \tag{8}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator, ν_0 is the viscosity coefficient (in what follows, a subscript 0 will denote bare parameters of the unrenormalized theory), $\nu_0 u_0 = c^2/(4\pi\sigma)$ represents the magnetic diffusivity (where we have already extracted dimensionless reciprocal magnetic Prandtl number u_0 for convenience), c is the speed of light, σ is the conductivity, $P(x)$ is the pressure, and $\mathbf{v} \equiv \mathbf{v}(x)$ is a solenoidal (owing to the incompressibility) velocity field. Thus, both $\mathbf{v}$ and $\mathbf{b}$ are divergence-free vector fields: $\boldsymbol{\partial} \cdot \mathbf{v} = \boldsymbol{\partial} \cdot \mathbf{b} = 0$.

A transverse Gaussian random noise $\mathbf{f}^{\mathbf{b}} = \mathbf{f}^{\mathbf{b}}(x)$ with zero mean and the correlation function,

$$D^b_{ij} \equiv \langle f^b_i(x) f^b_j(0) \rangle = \delta(t) C_{ij}(|\mathbf{x}|/L'), \tag{9}$$

represents the source of the fluctuation of the magnetic field $\mathbf{b}$ and maintains the steady state of the system. Here, L' is an integral scale related to the corresponding stirring, and C_{ij} is a function finite in the limit $L' \to \infty$. In what follows, the detailed form of the function C_{ij} is not important, the only condition which must be satisfied is that C_{ij} decreases rapidly for $|\mathbf{x}| \gg L'$. If C_{ij} depends on the direction of the vector $\mathbf{x}$ and not only on its modulus $r = |\mathbf{x}|$ then it can be considered as a source of large-scale anisotropy. In a more realistic formulation, the noise can be replaced, e.g., by the term $(\mathbf{B} \cdot \boldsymbol{\partial})\mathbf{v}$, where $\mathbf{B}$ is a constant large-scale magnetic field, the source of anisotropy (see, e.g., [24]).

On the other hand, the transverse random force per unit mass $\mathbf{f}^{\mathbf{v}} = \mathbf{f}^{\mathbf{v}}(x)$ in (8) simulates the energy pumping into the system on large scales. We assume that its statistics is Gaussian with zero mean and pair correlation function

$$D^v_{ij}(x; 0) = \langle f^v_i(x) f^v_j(0) \rangle = \delta(t) \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} D_0 k^{4-d-2\varepsilon} P_{ij}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{10}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ is the ordinary transverse projector, d denotes the spatial dimension of the system, $D_0 \equiv g_0 \nu_0^3 > 0$ is the positive amplitude, and the physical value of formally small parameter $0 < \varepsilon \leqslant 2$ is $\varepsilon = 2$. It plays an analogous role as the parameter $\epsilon = 4 - d$ in the theory of critical behavior and the introduced parameter g_0 plays the role of the coupling constant of the model. In addition, g_0 is a formal small parameter of the ordinary perturbation theory and it is related to the characteristic ultraviolet (UV) momentum scale Λ (or inner length $l \sim \Lambda^{-1}$) by the following relation:

$$g_0 \simeq \Lambda^{2\varepsilon}. \tag{11}$$

The correlation function (10) is chosen in the form which, on one hand, is suitable for the description of the real infrared energy pumping to the system (for $\varepsilon \to 2$ the function $D_0 k^{4-d-2\varepsilon}$ is proportional to $\delta(\mathbf{k})$ for appropriate choice of the amplitude factor D_0, which corresponds to the injection of energy to the system through interaction with the largest turbulent eddies) and, on the other hand, its powerlike form gives the possibility of applying the RG technique for the analysis of the problem [8, 11, 12, 38].

The integration in (10) is restricted by the condition $k \equiv |\mathbf{k}| > m$, where $m = 1/L$ is another integral scale. It provides the necessary infrared regularization. Such form of regularization is the convenient choice from a calculational point of view, although, a precise form of regularization is not essential. Besides, in what follows, we shall always assume that $L' \gg L$.

The most important feature of the stochastic model (7)–(10) is that the model is Galilean invariant as a consequence of the fact that the Gaussian distribution of the random force (10) is delta-correlated in time. It means that the corresponding perturbative expansions in the nonlinearities of the model are Galilean covariant, i.e., Ward identities, which represent exact relations between the correlation functions given by the Galilean symmetry, are held in all orders. Therefore, because the renormalization procedure does not disturb the Galilean symmetry, these exact relations will be also valid for perturbation expansions obtained by the RG and OPE. As a consequence, the Galilean invariant quantities as equal-time correlation or structure functions are not influenced by the sweeping which becomes important for $\varepsilon \geqslant 3/2$ for the present model [39]. Mathematically it means that the operators built of velocity field and its temporal derivatives do not contribute to the OPE for Galilean invariant correlation functions [8, 11, 12, 38]. On the other hand, physically it means that the sweeping by large-scale eddies does not influence the relative motion of the fluid or advected quantities within the inertial interval as was shown in [40] for the model (8), (10) for ε as high as $\varepsilon = 7/4$.

The field theoretic formulation of the present problem is based on the well-known theorem [41] which asserts that the stochastic problem (7)–(10) is equivalent to the field theoretic model of the doubled set of fields $\Phi = \{\mathbf{v}, \mathbf{b}, \mathbf{v}', \mathbf{b}'\}$ with the following action functional

$$S(\Phi) = \frac{1}{2}\int \mathrm{d}t_1\, \mathrm{d}^d\mathbf{x_1}\, \mathrm{d}t_2\, \mathrm{d}^d\mathbf{x_2}\big[v_i'(x_1)D_{ij}^v(x_1;x_2)v_j'(x_2) + b_i'(x_1)D_{ij}^b(x_1;x_2)b_j'(x_2)\big]$$
$$+\int \mathrm{d}t\,\mathrm{d}^d\mathbf{x}\{\mathbf{v}'[-\partial_t + \nu_0\triangle - (\mathbf{v}\cdot\boldsymbol{\partial})]\mathbf{v}$$
$$+\mathbf{b}'[-\partial_t\mathbf{b} + \nu_0 u_0\triangle\mathbf{b} - (\mathbf{v}\cdot\boldsymbol{\partial})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\partial})\mathbf{v}]\}, \tag{12}$$

where $x_i = (t_i, \mathbf{x_i})$, $i = 1, 2$, $\mathbf{v}'$ and $\mathbf{b}'$ are auxiliary transverse fields and D_{ij}^b, D_{ij}^v are given in (9) and (10), respectively, and required summations over dummy indices are assumed.

As a result of the fact that the auxiliary vector field $\mathbf{v}'(x)$ is also transverse, i.e., $\partial_i v_i' = 0$, one can omit the pressure term in (12). The corresponding term has the following form

$$\int \mathrm{d}t\,\mathrm{d}^d\mathbf{x}(\mathbf{v}'\cdot\boldsymbol{\partial})P,$$

and after the integration by parts it is evident that it vanishes, namely:

$$\int \mathrm{d}t\,\mathrm{d}^d\mathbf{x}(\mathbf{v}'\cdot\boldsymbol{\partial})P = -\int \mathrm{d}t\,\mathrm{d}^d\mathbf{x}P(\boldsymbol{\partial}\cdot\mathbf{v}') = 0.$$

Model (12) corresponds to a standard Feynman diagrammatic technique with the nonzero bare propagators $\langle b_i b_j'\rangle_0 = \langle b_j' b_i\rangle_0^*$, $\langle v_i v_j'\rangle_0 = \langle v_j' v_i\rangle_0^*$, $\langle b_i b_j\rangle_0$, and $\langle v_i v_j\rangle_0$ (in the frequency–momentum representation)

$$\langle b_i(\omega,\mathbf{k})b_j'(\omega,\mathbf{k})\rangle_0 = \frac{P_{ij}(\mathbf{k})}{-\mathrm{i}\omega + \nu_0 u_0 k^2}, \tag{13}$$

$$\langle v_i(\omega,\mathbf{k})v_j'(\omega,\mathbf{k})\rangle_0 = \frac{P_{ij}(\mathbf{k})}{-\mathrm{i}\omega + \nu_0 k^2}, \tag{14}$$

$$\langle b_i(\omega,\mathbf{k})b_j(\omega,\mathbf{k})\rangle_0 = \frac{C_{ij}(\mathbf{k})}{|-\mathrm{i}\omega + \nu_0 u_0 k^2|^2}, \tag{15}$$

$$\langle v_i(\omega,\mathbf{k})v_j(\omega,\mathbf{k})\rangle_0 = \frac{D_0 k^{4-d-2\varepsilon}P_{ij}(\mathbf{k})}{|-\mathrm{i}\omega + \nu_0 k^2|^2}, \tag{16}$$

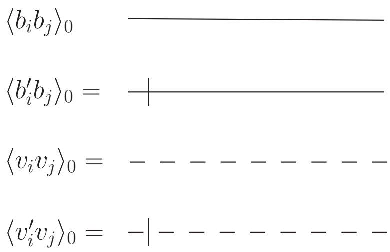

Figure 1. Graphical representation of the propagators of the model.

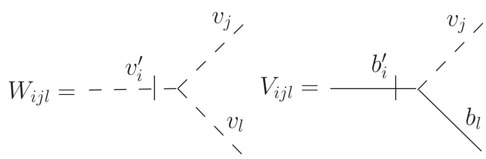

Figure 2. The triple (interaction) vertexes of the model. Momentum **k** is flowing into the vertexes via the auxiliary fields $\mathbf{b}'$ and $\mathbf{v}'$.

where $C_{ij}(\mathbf{k})$ is the Fourier transform of the function $C_{ij}(\mathbf{r}/L')$ from (9). In the Feynman diagrams these propagators are represented by lines which are shown in figure 1 (the end with a slash in the propagators $\langle b_i b_j' \rangle_0$ and $\langle v_i v_j' \rangle_0$ corresponds to the fields $\mathbf{b}'$ and $\mathbf{v}'$, respectively, and the end without a slash corresponds to the fields **b** and **v**, respectively). The triple vertices (or interaction vertices) $b_i'(-v_j \partial_j b_i + b_j \partial_j v_i) = b_i' v_j V_{ijl} b_l$ and $-v_i' v_j \partial_j v_i = v_i' v_j W_{ijl} v_l/2$, where $V_{ijl} = \mathrm{i}(k_j \delta_{il} - k_l \delta_{ij})$ and $W_{ijl} = \mathrm{i}(k_l \delta_{ij} + k_j \delta_{il})$ (in the momentum–frequency representation) are present in figure 2, where momentum **k** is flowing into the vertices via the auxiliary fields $\mathbf{b}'$ and $\mathbf{v}'$, respectively.

The formulation of the problem through the action functional (12) replaces the statistical averages of random quantities in the stochastic problem defined by equations (7)–(10) with equivalent functional averages with weight $\exp S(\Phi)$. The generating functionals of the total Green's functions $G(A)$ and connected Green's functions $W(A)$ are then defined by the functional integral

$$G(A) = \mathrm{e}^{W(A)} = \int \mathcal{D}\Phi \, \mathrm{e}^{S(\Phi) + A\Phi}, \tag{17}$$

where $A(x) = \{\mathbf{A}^{\mathbf{b}}, \mathbf{A}^{\mathbf{b}'}, \mathbf{A}^{\mathbf{v}}, \mathbf{A}^{\mathbf{v}'}\}$ represents a set of arbitrary sources for the set of fields Φ, $\mathcal{D}\Phi \equiv \mathcal{D}\mathbf{b}\mathcal{D}\mathbf{b}'\mathcal{D}\mathbf{v}\mathcal{D}\mathbf{v}'$ denotes the measure of functional integration, and the linear form $A\Phi$ is defined as

$$A\Phi = \int \mathrm{d}x \big[A_i^b(x) b_i(x) + A_i^{b'}(x) b_i'(x) + A_i^v(x) v_i(x) + A_i^{v'}(x) v_i'(x) \big]. \tag{18}$$

3. Renormalization group analysis and scaling regime

The information about possible UV divergences in the model can be found by the standard analysis of canonical dimensions (see, e.g., [8, 7]). The dynamical model (12) belongs to the class of the so-called two-scale models [8, 11, 12], i.e., to the class of models for which the canonical dimension of some quantity F is given by two numbers, namely, the momentum

Table 1. Canonical dimensions of the fields and parameters of the model under consideration.

F	$\mathbf{v}$	$\mathbf{v}'$	$\mathbf{b}$	$\mathbf{b}'$	m, Λ, μ	ν_0, ν	g_0	g, u_0, u
d_F^k	-1	$d+1$	0	d	1	-2	2ε	0
d_F^ω	1	-1	$-1/2$	$1/2$	0	1	0	0
d_F	1	$d-1$	-1	$d+1$	1	0	2ε	0

dimension d_F^k and the frequency dimension d_F^ω. To find the dimensions of all quantities it is convenient to use the standard normalization conditions $d_k^k = -d_x^k = 1, d_\omega^\omega = -d_t^\omega = 1$, $d_k^\omega = d_x^\omega = d_\omega^k = d_t^k = 0$, and the requirement that each term of the action functional must be dimensionless separately with respect to the momentum and frequency dimensions. The total canonical dimension d_F is then defined as $d_F = d_F^k + 2d_F^\omega$ (it is related to the fact that $\partial_t \propto \partial^2$ in the free action (12) with choice of zero canonical dimensions for ν_0 and u_0). In the framework of the theory of renormalization the total canonical dimension in dynamical models plays the same role as the momentum dimension does in static models.

The canonical dimensions of our model are present in table 1, where also the canonical dimensions of the renormalized parameters are shown.

The model (12) is logarithmic at $\varepsilon = 0$ (the coupling constants g_0 is dimensionless); therefore, in the framework of the minimal subtraction (MS) scheme [7], which is always used in what follows, possible UV divergences in the correlation functions have the form of poles in ε. It is well known that the superficial divergences can be present only in the 1-irreducible Green's functions for which the corresponding total canonical dimensions are a nonnegative integer. Detail analysis shows that for spatial dimensions $d > 2$ superficially divergent functions of our model are only functions $\langle \mathbf{v}'\mathbf{v}\rangle_{1-\mathrm{ir}}$ and $\langle \mathbf{b}'\mathbf{b}\rangle_{1-\mathrm{ir}}$ and, in this case, the action (12) has all necessary tensor structures to remove divergences multiplicatively (see, e.g., [7, 8]). All divergences can be removed by the counterterms of the forms $\mathbf{v}'\Delta\mathbf{v}$ and $\mathbf{b}'\Delta\mathbf{b}$ what can be explicitly expressed in the multiplicative renormalization of the parameters g_0, u_0 and ν_0 in the form

$$\nu_0 = \nu Z_\nu, \qquad g_0 = g\mu^{2\varepsilon} Z_g, \qquad u_0 = u Z_u, \tag{19}$$

where the dimensionless parameters g, u and ν are the renormalized counterparts of the corresponding bare ones, μ is the renormalization mass (a scale-setting parameter), an artifact of the dimensional regularization. Quantities $Z_i = Z_i(g, u; d; \varepsilon)$ are the so-called renormalization constants and they contain poles in ε.

The renormalized action functional has the following form:

$$\begin{aligned} S^R(\Phi) = \frac{1}{2}\int & \mathrm{d}t_1\, \mathrm{d}^d\mathbf{x_1}\, \mathrm{d}t_2\, \mathrm{d}^d\mathbf{x_2}\big[v_i'(x_1) D_{ij}^v(x_1; x_2) v_j'(x_2) + b_i'(x_1) D_{ij}^b(x_1; x_2) b_j'(x_2)\big] \\ & + \int \mathrm{d}t\, \mathrm{d}^d\mathbf{x}\{\mathbf{v}'[-\partial_t + \nu Z_1 \triangle - (\mathbf{v}\cdot\boldsymbol{\partial})]\mathbf{v} \\ & + \mathbf{b}'[-\partial_t \mathbf{b} + \nu u Z_2 \triangle \mathbf{b} - (\mathbf{v}\cdot\boldsymbol{\partial})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\partial})\mathbf{v}]\}. \end{aligned} \tag{20}$$

By comparison of the renormalized action (20) with definitions of the renormalization constants $Z_i, i = g, u, \nu$, which are given in (19), we come to the relations among them:

$$Z_\nu = Z_1, \qquad Z_g = Z_1^{-3}, \qquad Z_u = Z_2 Z_1^{-1}. \tag{21}$$

The renormalization constants Z_1 and Z_2 are determined by the requirement that the one-particle irreducible Green's functions $\langle \mathbf{v}'\mathbf{v}\rangle_{1-\mathrm{ir}}$ and $\langle \mathbf{b}'\mathbf{b}\rangle_{1-\mathrm{ir}}$ must be UV finite when are written in the renormalized variables, i.e., they have no singularities in the limit $\varepsilon \to 0$. On the

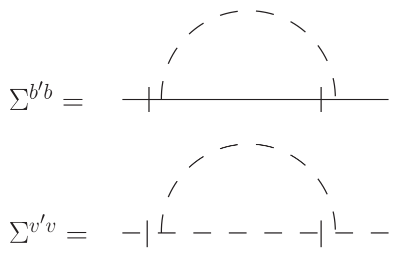

Figure 3. The one-loop diagrams that contribute to the self-energy operators $\Sigma_{v'v}$ and $\Sigma_{b'b}$.

other hand, these one-particle irreducible Green's functions are related to the corresponding self-energy operators $\Sigma^{v'v}$ and $\Sigma^{b'b}$, which are expressed via Feynman diagrams, by the Dyson equations. In frequency–momentum representation they can be written in the following form

$$\langle v_i' v_j \rangle_{1-ir} = (-\mathrm{i}\omega + \nu_0 p^2 - \Sigma^{v'v}(\omega, p)) P_{ij}(\mathbf{p}), \tag{22}$$

$$\langle b_i' b_j \rangle_{1-ir} = (-\mathrm{i}\omega + \nu_0 u_0 p^2 - \Sigma^{b'b}(\omega, p)) P_{ij}(\mathbf{p}). \tag{23}$$

Thus, Z_1 and Z_2 are found from the requirement that the UV divergences be canceled in (22) and (23) after the substitution $e_0 = e\mu^{d_e} Z_e$ for $e = \{g, u, \nu\}$. This determines Z_1 and Z_2 up to an UV finite contribution, which is fixed by the choice of the renormalization scheme. In the MS scheme all the renormalization constants have the form: 1 + *poles in* ε. In one-loop approximation the self-energy operators $\Sigma^{v'v}$ and $\Sigma^{b'b}$ are given by Feynman diagrams which are shown in figure 3 and their explicit analytical form suitable for the representation as shown in (22) and (23) is given as follows

$$\Sigma^{v'v}(p) = -\frac{S_d}{(2\pi)^d} \frac{g\nu p^2}{2\varepsilon} \frac{d-1}{4(d+2)}, \tag{24}$$

$$\Sigma^{b'b}(p) = -\frac{S_d}{(2\pi)^d} \frac{g\nu p^2}{2\varepsilon} \frac{d-1}{2d(u+1)}, \tag{25}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface of the d-dimensional unit sphere.

Thus, the renormalization constants Z_1 and Z_2 are given as follows

$$Z_1 = 1 - \frac{\bar{g}}{2\varepsilon} \frac{d-1}{4(d+2)}, \tag{26}$$

$$Z_2 = 1 - \frac{\bar{g}}{2\varepsilon} \frac{d-1}{2du(u+1)}, \tag{27}$$

where we have introduced suitable notation $\bar{g} = gS_d/(2\pi)^d$.

The relation $S(\Phi, e_0) = S^R(\Phi, e, \mu)$, where e_0 stands for the complete set of bare parameters and e stands for the renormalized ones, leads to the relation $W(A, e_0) = W^R(A, e, \mu)$ for the generating functional of connected Green's functions. By application of the operator $\mathcal{D}_\mu \equiv \mu\partial_\mu$ at fixed bare parameters e_0 on both sides of the last equation one obtains the basic RG differential equation

$$\mathcal{D}_{\mathrm{RG}} W^R(A, e, \mu) = 0, \tag{28}$$

where the explicit form of the operator $\mathcal{D}_{RG}$ is as follows

$$\mathcal{D}_{RG} = \mathcal{D}_\mu + \beta_g(g)\partial_g + \beta_u(g,u)\partial_u - \gamma_\nu(g)\mathcal{D}_\nu, \tag{29}$$

where we denote $\mathcal{D}_x \equiv x\partial_x$ for any variable x and the RG functions (the β and γ functions) are given by the well-known definitions

$$\beta_g \equiv \mathcal{D}_\mu g = g(-2\varepsilon + 3\gamma_1), \tag{30}$$

$$\beta_u \equiv \mathcal{D}_\mu u = u(\gamma_1 - \gamma_2), \tag{31}$$

$$\gamma_i \equiv \mathcal{D}_\mu \ln Z_i, \qquad i = 1, 2, \tag{32}$$

$$\gamma_\nu = \gamma_1, \tag{33}$$

where relations among renormalization constants (21) were used and Z_1 and Z_2 are given in (26) and (27), respectively. Thus, the explicit form of the anomalous dimensions γ_1 and γ_2 is

$$\gamma_1 = \bar{g}\frac{d-1}{4(d+2)}, \qquad \gamma_2 = \bar{g}\frac{d-1}{2du(u+1)}. \tag{34}$$

Standardly, possible scaling regimes of a renormalized model are related to the existence of IR stable fixed points of the corresponding system of the RG equations [7, 8]. The fixed point of the RG equations is defined by the β functions, more precisely, by requirement of their vanishing. In our case, the coordinates g_*, u_* of the possible fixed points are given by the system of two equations

$$\beta_g(g_*, u_*) = \beta_u(g_*, u_*) = 0, \tag{35}$$

with β_g and β_u given in (30) and (31). The nontrivial positive solution of this system of equations is given as follows

$$\bar{g}_* = \frac{8}{3}\frac{d+2}{d-1}\varepsilon, \qquad u_* = \frac{1}{2}\left(-1 + \sqrt{\frac{9d+16}{d}}\right). \tag{36}$$

On the other hand, the IR stability of the fixed point is given by the condition that the real parts of the eigenvalues of the matrix of the first derivatives

$$\Omega_{ij} = \begin{pmatrix} \partial\beta_g/\partial g & \partial\beta_g/\partial u \\ \partial\beta_u/\partial g & \partial\beta_u/\partial u \end{pmatrix} \tag{37}$$

must be positive. In our case, the eigenvalues of the matrix (37) taken at the fixed point (36) are

$$\lambda_1 = 2\varepsilon, \qquad \lambda_2 = \bar{g}_*\frac{d-1}{2d}\frac{1+2u_*}{u_*(u_*+1)^2}. \tag{38}$$

As one can see, both eigenvalues are positive for $\varepsilon > 0$ and for positive values of g_* and u_*.

The above one-loop expressions for the coordinates of the fixed point (36) in our vector model are the same as the one-loop expressions for the fixed point in the problem of passive scalar advection by the Navier–Stokes velocity field [36] (see also the pioneer paper [42]). Thus, although the mathematical structure of the vector model is much more complicated than the corresponding model of scalar advection, nevertheless their IR scaling behavior is the same. We shall see in the next section that the same situation arises also for the anomalous behavior of the models at least at the one-loop level approximation.

Existence of the stable IR fixed point means that the correlation functions of the model exhibit scaling behavior with given critical dimensions in the IR range. The issue of interest is especially multiplicatively renormalizable equal-time two-point quantities $G(r)$. Examples of such quantities are the equal-time structure functions in the inertial interval as they were defined in (1). The IR scaling behavior of the function $G(r)$ (for $r/l \gg 1$ and any fixed r/L)

$$G(r) \simeq \nu_0^{d_G^\omega} l^{-d_G} (r/l)^{-\Delta_G} R(r/L) \tag{39}$$

is related to the existence of IR stable fixed point of the RG equations (36). In (39) d_G^ω and d_G are the corresponding canonical dimensions of the function G (the canonical dimensions of the model are given in table 1), the UV momentum scale $\Lambda = 1/l$ is defined in (11), $R(r/L)$ is a scaling function, which, as was already mentioned in introduction, cannot be determined by the RG equations (see, e.g., [8]), and Δ_G is the critical dimension defined as

$$\Delta_G = d_G^k + \Delta_\omega d_G^\omega + \gamma_G^*. \tag{40}$$

Here γ_G^* is the fixed point value of the anomalous dimension $\gamma_G \equiv \mu \partial_\mu \ln Z_G$, where Z_G is the renormalization constant of the multiplicatively renormalizable quantity G, i.e., $G = Z_G G^R$ [8], and $\Delta_\omega = 2 - \gamma_\nu^*$ is the critical dimension of the frequency with $\gamma_\nu^* = \gamma_1^*$ which is defined in (34) and γ_1^* means that γ_1 is taken at the fixed point. From (30) and (35) one immediately finds that $\gamma_\nu^* \equiv \gamma_1^* = 2\varepsilon/3$. It is the exact one-loop result, i.e., no higher-loop corrections to the γ_ν^* exist. It means that the critical dimension of frequency is also known exactly, namely, $\Delta_\omega = 2(1 - \varepsilon/3)$, as well as the critical dimensions of the fields:

$$\Delta_{\mathbf{v}} = 1 - \frac{2\varepsilon}{3}, \qquad \Delta_{\mathbf{v}'} = d - 1 + \frac{2\varepsilon}{3}, \tag{41}$$

$$\Delta_{\mathbf{b}} = -1 + \frac{\varepsilon}{3}, \qquad \Delta_{\mathbf{b}'} = d + 1 - \frac{\varepsilon}{3}. \tag{42}$$

An example of interesting equal-time quantities built of the magnetic field $\mathbf{b}$ are the equal-time structure functions defined in analogy with (1)

$$S_N(r) = \langle [b_r(t, \mathbf{x}) - b_r(t, \mathbf{x}')]^N \rangle. \qquad r = |\mathbf{x} - \mathbf{x}'|. \tag{43}$$

They are important tools in the analysis of the MHD turbulence within the inertial range $l \ll r \ll L$ but, in what follows, we shall concentrate on simpler equal-time quantities, namely, on the equal-time two-point correlation functions

$$B_{N-m,m}(r) \equiv \left\langle b_r^{N-m}(t, \mathbf{x}) b_r^m(t, \mathbf{x}') \right\rangle, \qquad r = |\mathbf{x} - \mathbf{x}'| \tag{44}$$

i.e., we shall analyze the equal-time correlation functions built of two composite operators $b_r^{N-m}(t, \mathbf{x})$ and $b_r^m(t, \mathbf{x})$. The reason is twofold: first, the structure functions (43) are given by linear combinations of the correlation functions (44); hence the scaling behavior of the structure functions (43) emerges from the scaling behavior of the correlation functions (44) also as a linear combination. Second, there is no special need to investigate the structure functions (which are more complex quantities) instead of their building blocks, the correlation functions (44), as a result of the fact that contrary to the passive scalar advection by incompressible velocity field the stochastic equation for the vector field $\mathbf{b}$ (7) is not invariant under the shift $\mathbf{b} \to \mathbf{b} + \mathbf{b_0}$, where $\mathbf{b_0}$ is a constant vector.

Applying the general scaling representation for the equal-time quantity $G(r)$ given in (39) with the critical dimension Δ_G given in (40) to the correlation functions (44) one comes to the result

$$B_{N-m,m}(r) \simeq \nu_0^{-N/2} l^N (r/l)^{N(1-\varepsilon/3) - \gamma_{N-m}^* - \gamma_m^*} R_{N,m}(r/L), \tag{45}$$

where γ^*_{N-m} and γ^*_m are the fixed point values of the anomalous dimensions of the composite operators b_r^{N-m} and b_r^m, respectively and the scaling functions $R_{N,m}(r/L)$ remain unknown within the standard RG analysis.

On the other hand, the behavior of the functions $R_{N,m}(r/L)$ in the limit $r/L \to 0$ can be estimated using the OPE, which leads to the following asymptotic form of the scaling functions in the given limit:

$$R_{N,m}(r/L) = \sum_i C_{F_i}(r/L)(r/L)^{\Delta_{F_i}}, \tag{46}$$

where $C_F(r/L)$ are coefficients regular in r/L and the summation is implied over all possible renormalized composite operators F_i allowed by symmetry with critical dimensions Δ_{F_i}. In the case under consideration the leading contribution (with the smallest critical dimensions) into the OPE (46) is given by the operators F_i having the form

$$F[N, p] = b_{i_1} \cdots b_{i_p} (\mathbf{b} \cdot \mathbf{b})^n, \qquad N = 2n + p, \tag{47}$$

which allow also to take into account effects of anisotropy (see the next section). In purely isotropic situation the set of operators (47) is reduced to the operator $F_N = (\mathbf{b} \cdot \mathbf{b})^{N/2}$. Thus, by using expression (40) for the critical dimension, the final asymptotic behavior of the correlation functions (44) will be given as follows

$$\begin{aligned} B_{N-m,m}(r) &\simeq \nu_0^{-N/2} L^N \left(\frac{l}{L}\right)^{N\varepsilon/3} \left(\frac{r}{l}\right)^{-\gamma^*_{N-m}-\gamma^*_m} \left(\frac{r}{L}\right)^{\gamma^*_N} \\ &\sim r^{-\gamma^*_{N-m}-\gamma^*_m+\gamma^*_N}, \end{aligned} \tag{48}$$

where γ^*_N are the anomalous dimensions of the composite operators $F_N = (\mathbf{b} \cdot \mathbf{b})^{N/2}$ (see the next section for details). For special cases $m = 0$ or $m = N$ the correlation functions $B_{N-m,m}(r)$ are reduced to the constants, namely,

$$B_{N,0} \equiv B_{0,N} \simeq \nu_0^{-N/2} L^N \left(\frac{l}{L}\right)^{N\varepsilon/3}. \tag{49}$$

On the other hand, in the anisotropic case, the asymptotic behavior of the correlation functions (44) will be driven by the set of critical dimensions which corresponds to the composite operators which are mixed during the renormalization and the leading contribution to the asymptotic behavior of the correlation functions will be given by the smallest critical dimensions. It is the main aim of the following section to analyze this issue in more details and to find the explicit form of the critical dimensions for needed composite operators (47) in the case with the presence of large-scale anisotropy in the one-loop approximation. It will lead to the explicit form for the inertial-range behavior of the single-time correlation functions $B_{N-m,m}$.

4. Renormalization and critical dimensions of composite operators and anomalous scaling

4.1. Operator product expansion

The OPE [7, 8, 11, 12] asserts that the equal-time product $F_1(x')F_2(x'')$ of two renormalized composite operators[1] at $\mathbf{x} = (\mathbf{x}' + \mathbf{x}'')/2 = \text{const}$ and $\mathbf{r} = \mathbf{x}' - \mathbf{x}'' \to 0$ can be written in the following form:

$$F_1(x')F_2(x'') = \sum_i C_i F_i(t, \mathbf{x}), \tag{50}$$

[1] By definition we use the term 'composite operator' for any local monomial or polynomial constructed from primary fields and their derivatives at a single point $x \equiv (t, \mathbf{x})$. Constructions $\theta^n(x)$ and $[\partial_i\theta(x)\partial_i\theta(x)]^n$ are typical examples.

where the summation is taken over all possible renormalized local composite operators F_i allowed by symmetry with definite critical dimensions Δ_{F_i}, and the functions C_i are the corresponding Wilson coefficients regular in L^{-2}. The renormalized correlation function $\langle F_1(x')F_2(x'')\rangle$ can now be found by averaging (50) with the weight $\exp S^R$ with S^R from (20). The quantities $\langle F_i\rangle$ appear on the right-hand side and their asymptotic behavior in the limit $L^{-1} \to 0$ is then found from the corresponding RG equations and has the form $\langle F_i\rangle \propto L^{-\Delta_{F_i}}$.

As was already mentioned in introduction, the specific feature of the turbulence models is the existence of operators with negative critical dimensions in the OPE (the so-called dangerous operators) which determine the IR behavior of the scaling functions and lead to their singular dependence on L when $r/L \to 0$.

Two-point correlation functions (44) are averages of products of composite operators $b_r^N(t,\mathbf{x})$ at two separate space points: $b_r^{N-m}(t,\mathbf{x})b_r^m(t,\mathbf{x}')$. Therefore, it is evident that the leading contribution to the OPE will be given by the closed set of operators generated by the operator $b_r^N(t,\mathbf{x})$ (it is given by the fact that not only do the operators which are present in the corresponding Taylor expansion enter into the OPE but also all possible operators that admix to them in renormalization, see the next subsection for details). In the isotropic case the principal role is played by the composite operators $F_N = F_{2n} = (\mathbf{b}\cdot\mathbf{b})^n$. On the other hand, in the model with the presence of uniaxial anisotropy (large- or small-scale) the leading contribution of the Taylor expansion is given by the tensor composite operators constructed solely of the fields $\mathbf{b}$ without derivatives given in (47), where p denotes the free vector indices.

4.2. *Composite operators $F[N,p]$: renormalization and critical dimensions*

Let us briefly discuss renormalization of the composite operators (47). The necessity of additional renormalization of the composite operators (47) is related to the fact that the coincidence of the field arguments in Green's functions containing them leads to additional UV divergences. These divergences must be removed by a special kind of renormalization procedure which can be found, e.g., in [7, 8], where their renormalization is studied in general. As for the renormalization of composite operators in the models of turbulence it is discussed in [11, 12]. Typically, the composite operators are mixed under renormalization. Therefore, let us briefly discuss this issue [8].

Let $F \equiv \{F_\alpha\}$ be a closed set of composite operators which are mixed only with each other in renormalization. Then the renormalization matrix $Z_F \equiv \{Z_{\alpha\beta}\}$ and the matrix of corresponding anomalous dimensions $\gamma_F \equiv \{\gamma_{\alpha\beta}\}$ for this set are given as follows

$$F_\alpha = \sum_\beta Z_{\alpha\beta}F_\beta^R, \qquad \gamma_F = Z_F^{-1}\mathcal{D}_\mu Z_F. \tag{51}$$

Renormalized composite operators are subject to the following RG differential equations

$$\left(\mathcal{D}_\mu + \sum_{i=g,u}\beta_i\partial_i - \gamma_\nu\mathcal{D}_\nu\right)F_\alpha^R = -\sum_\beta \gamma_{\alpha\beta}F_\beta^R, \tag{52}$$

which lead to the following matrix of critical dimensions $\Delta_F \equiv \{\Delta_{\alpha\beta}\}$

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \qquad \Delta_\omega = 2 - \gamma_\nu^*, \tag{53}$$

where d_F^k a d_F^ω are diagonal matrices of corresponding canonical dimensions and γ_F^* is the matrix of anomalous dimensions (51) taken at the fixed point. In the end, the critical dimensions of the set of operators $F \equiv \{F_\alpha\}$ are given by the eigenvalues of the matrix Δ_F. The so-called 'basis' operators that possess definite critical dimensions have the form

$$F_\alpha^{\rm bas} = \sum_\beta U_{\alpha\beta} F_\beta^R, \tag{54}$$

where the matrix $U_F = \{U_{\alpha\beta}\}$ is such that $\Delta'_F = U_F \Delta_F U_F^{-1}$ is diagonal.

As was discussed in the previous subsection, in what follows, the central role is played by the tensor composite operators given in (47). It is convenient to deal with the scalar operators obtained by contracting the tensors with the appropriate number of the uniaxial anisotropy unit vectors **n** [27],

$$F[N,p] = [\mathbf{n}\cdot\mathbf{b}]^p(\mathbf{b}\cdot\mathbf{b})^n, \qquad N = 2n+p, \tag{55}$$

As can be shown by direct analysis of the diagrams the composite operators (55) with different N are not mixed in renormalization, and therefore the corresponding renormalization matrix $Z_{[N,p][N',p']}$ is in fact block-diagonal, i.e., $Z_{[N,p][N',p']} = 0$ for $N' \neq N$ (see, e.g., [27] for details). Thus, in general, one can write

$$F[N,p] = \sum_{l=0}^{\lfloor N/2 \rfloor} Z_{[N,p][N,N-2l]} F^R[N, N-2l], \tag{56}$$

where $\lfloor N/2 \rfloor$ means the integer part of the $N/2$. Therefore, each block of renormalization constants with given N is an $(\lfloor N/2 \rfloor + 1) \times (\lfloor N/2 \rfloor + 1)$ matrix. Of course, the matrix of critical dimensions (53), whose eigenvalues at IR stable fixed point are the critical dimensions $\Delta[N,p]$ of the set of operators $F[N,p]$, has also dimension $(\lfloor N/2 \rfloor + 1) \times (\lfloor N/2 \rfloor + 1)$.

Equation (56) represents a general situation but, as we shall see, in our model with the presence of large-scale anisotropy the elements $Z_{[N,p][N,p']}$ vanish for $p < p'$; thus the block $Z_{[N,p][N,p']}$ is in fact triangular along with the corresponding blocks of the matrices U_F and Δ_F from (54) and (53). Therefore, the critical dimensions of the basis operators will be directly given by the diagonal elements of the corresponding matrix.

Let us turn to the calculation of the renormalization constants $Z_{[N,p][N,p']}$ in the one-loop approximation. We shall proceed as in [24, 27, 29]. If we denote as $\Gamma(x;\mathbf{b})$ the generating functional of the 1-irreducible Green's functions with one composite operator $F[N,p]$ from (55) and any number of fields **b** then we are interested in the Nth term of the expansion of $\Gamma(x;\mathbf{b})$ in **b**, which we denote $\Gamma_N(x;\mathbf{b})$. It has the following form [24]

$$\Gamma_N(x;\mathbf{b}) = \frac{1}{N!}\int {\rm d}x_1 \cdots \int {\rm d}x_N b_{i_1}(x_1)\cdots b_{i_N}(x_N) \times \langle F[N,p](x) b_{i_1}(x_1)\cdots b_{i_N}(x_N)\rangle_{1-{\rm ir}}, \tag{57}$$

where summations over dummy indexes are understood. In the one-loop approximation it is given as

$$\Gamma_N = F[N,p] + \Gamma^{(1)}, \tag{58}$$

where $\Gamma^{(1)}$ is given by the analytical calculation of the diagram in figure 4 and the first term in (58) represents 'tree' approximation (55).

The black circle with two attached lines in the diagram in figure 4 denotes the variational derivative $V_{ij}(x;x_1,x_2) \equiv \delta^2 F[N,p]/\delta b_i(x_1)\delta b_j(x_2)$, where the second variation makes needed combinatorics, namely, the operator $F[N,p]$ contains N components of the field **b** and one must take two of them (in all possible ways) to construct the one-loop diagram as it is shown in figure 4. It can be represented in the following convenient form [27]

$$V_{ij}(x;x_1,x_2) = \delta(x-x_1)\delta(x-x_2)\frac{\partial^2}{\partial a_i \partial a_j}[(\mathbf{na})^p(a^2)^n], \tag{59}$$

$\Gamma^{(1)} = \frac{1}{2}$

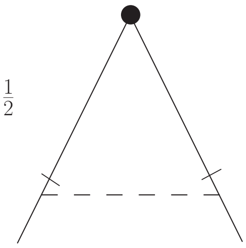

Figure 4. Graphical representation of the one-loop correction to Γ_N in equation (58).

where a constant vector a_k will be substituted with $b_k(x)$ after the differentiation. Analytical form of the diagram in figure 4 (without the symmetry factor 1/2) is the following:

$$\int \mathrm{d}x_1 \cdots \int \mathrm{d}x_4 V_{ij}(x; x_1, x_2) \langle b_i(x_1) b'_k(x_3) \rangle_0 \\ \times \langle b_j(x_2) b'_l(x_4) \rangle_0 \langle \partial_p v_k(x_3) \partial_q v_l(x_4) \rangle_0 b_p(x_3) b_q(x_4), \tag{60}$$

where the bare propagators are given in (13) and (16). The derivatives of the velocity fields are related to the second term of the ordinary vertex factors $b'_i(-v_j \partial_j b_i + b_j \partial_j v_i)$ shown in figure 2. The first terms of the vertices are omitted because they are proportional to the derivatives of the field $\mathbf{b}$ and we know that the UV divergent part of the diagram is proportional to the N factors $\mathbf{b}$ without derivatives.

By setting the external momentum in the integrand equal to zero because the UV divergent part of the diagram in figure 4 is free of external momentum and after some simple manipulations the UV divergent part can be written in the following compact form:

$$a_k a_l \frac{\partial^2}{\partial a_i \partial a_j} [(\mathbf{na})^p (a^2)^n] X_{ij,kl}, \tag{61}$$

where $X_{ij,kl}$ in the momentum–frequency representation (suitable for the further calculations), after simple integration over the frequency, has the form

$$X_{ij,kl} = \frac{g_0}{2u_0(1+u_0)} \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \frac{q_k q_l}{k^{2+d+2\varepsilon}} P_{ij}(\mathbf{q}). \tag{62}$$

After integration over the momentum $\mathbf{q}$ and simple tensor manipulations one obtains the following result for the diagram shown in figure 4 (the symmetric factor 1/2 is included)

$$\frac{\bar{g}}{2\varepsilon} \left(\frac{\mu}{m}\right)^{2\varepsilon} \frac{\{Q_1 F[N, p-2] + Q_2 F[N, p]\}}{4d(d+2)u(1+u)}, \tag{63}$$

where we have substituted the unrenormalized quantities with the renormalized one, a_i have been replaced with the components of the field b_i (thus they again form the operators $F[N, q]$, with $q = p - 2$ and p), and the coefficients Q_i, $i = 1, 2$ are

$$Q_1 = p(p-1)(d+1), \tag{64}$$

$$Q_2 = 4n(d-1)(n+p-1) + 2n(d(d+1)-2) - 2p(p-1). \tag{65}$$

Now, using the standard renormalization procedure the renormalization constants $Z_{[N,p][N,p']}$ defined in (56) are found from the requirement that function (58) be UV finite (contains no poles in ε) when written in renormalized variables and with the replacement $F[N, p] \to F^R[N, p]$, namely,

$$Z_{[N,p][N,p-2]} = \frac{\bar{g}}{8d(d+2)u(u+1)\varepsilon} Q_1, \tag{66}$$

$$Z_{[N,p][N,p]} = 1 + \frac{\bar{g}}{8d(d+2)u(u+1)\varepsilon} Q_2, \tag{67}$$

with coefficients Q_i given in (64) and (65). Using the definition of the matrix of anomalous dimensions $\gamma_{[N,p][N',p']}$ as given in (51) we obtain the following result

$$\gamma_{[N,p][N,p-2]} = -\frac{\bar{g}}{4d(d+2)u(u+1)} Q_1, \tag{68}$$

$$\gamma_{[N,p][N,p]} = -\frac{\bar{g}}{4d(d+2)u(u+1)} Q_2, \tag{69}$$

and the desired matrix of critical dimensions (53) has the form

$$\Delta_{[N,p][N,p']} = -N\left(1 - \frac{\varepsilon}{3}\right) + \gamma^*_{[N,p][N,p']}, \tag{70}$$

where the asterisk means that the quantities are taken at the corresponding fixed point (see section 4).

In the end, the critical dimensions $\Delta[N, p]$ are given by the eigenvalues of the matrix (70) but, because the matrix $\gamma_{[N,p][N,p']}$ is triangular in our case, the eigenvalues of the matrix of critical dimensions $\Delta_{[N,p][N,p']}$ are given directly by the diagonal elements $\gamma_{[N,p]} \equiv \gamma_{[N,p][N,p]}$, namely,

$$\Delta_{[N,p]} = -N\left(1 - \frac{\varepsilon}{3}\right) + \gamma^*_{[N,p]}, \tag{71}$$

where $\gamma_{[N,p]}$ is given in (69) and taken at the fixed point.

4.3. Anomalous scaling of the correlation functions in one-loop approximation

Thus, now when one applies the above results given in (65), (69) and (71) in representation (45) the final asymptotic inertial range expression (48) for the single-time correlation functions (44) is obtained where, in our anisotropic case (for more details see, e.g., [24]), the corresponding fixed point values of the anomalous dimensions γ^*_N are replaced by the anomalous dimensions $\gamma^*_{[N,p]}$ with value of p which corresponds to the minimal value of $\gamma^*_{[N,p]}$, namely,

$$B_{N-m,m}(r) \sim r^{-\gamma^*_{[N-m,p_{N-m}]} - \gamma^*_{[m,p_m]} + \gamma^*_{[N,p_N]}}, \tag{72}$$

where we have denoted as p_q the value of p for which the corresponding $\gamma^*_{[q,p]}$ have a minimal value.

First of all, let us analyze the general expression for anomalous dimensions $\gamma^*_{[N,p]}$ for the fixed point values $\bar{g}_*$ and u_* as are defined in (36). It leads to the result

$$\gamma^*_{[N,p]} = -\frac{\varepsilon}{3(d-1)(d+2)} Q_2, \tag{73}$$

where Q_2 is given in (65) and we recall that $N = 2n + p$. Up to normalization, the expression (73) is the same as the one loop result obtained within the kinematic MHD Kazantzev–Kraichnan model analyzed in [24] (see the corresponding (6.17) and (6.18) in [24]). After simple analysis one finds that the following important hierarchies are obeyed by anomalous dimensions $\gamma^*_{N,p}$ (of course, the same is also true for critical dimensions $\Delta_{[N,p]}$):

$$\gamma^*_{[N,p]} < \gamma^*_{[N,p']}, \qquad p < p', \tag{74}$$

$$\gamma^*_{[N,0]} < \gamma^*_{[N',0]}, \qquad N > N', \tag{75}$$

$$\gamma^*_{[N,1]} < \gamma^*_{[N',1]}, \qquad N > N', \tag{76}$$

where (75) is valid for even values of N and N' and (76) is valid for odd values of N and N', respectively.

Using the hierarchy relations (74)–(76) one can write the following formulae for asymptotic behavior of the correlation functions $B_{N-m,m}(r)$ as functions of N and m:

$$B_{N-m,m}(r) \sim r^{\gamma^*_{[N,0]} - \gamma^*_{[N-m,0]} - \gamma^*_{[m,0]}}, \tag{77}$$

which holds for even values of N and m,

$$B_{N-m,m}(r) \sim r^{\gamma^*_{[N,0]} - \gamma^*_{[N-m,1]} - \gamma^*_{[m,1]}}, \tag{78}$$

which is valid for even value of N and odd value of m, and

$$B_{N-m,m}(r) \sim r^{\gamma^*_{[N,1]} - \gamma^*_{[N-m,0]} - \gamma^*_{[m,1]}}, \tag{79}$$

for odd values of N and m. The fourth possibility, namely, odd N and even m is equivalent to the last case.

In the end, using the explicit expression for the fixed point value of the anomalous dimension $\gamma_{[N,p]}$ as given in (73) one obtains the explicit form for the asymptotic behavior of the correlation functions $B_{N-m,m}(r)$, namely,

$$B_{N-m,m}(r) \sim r^{-\frac{2\varepsilon}{3(d+2)}A}, \tag{80}$$

where

$$A = \left\{ \begin{array}{ll} (N-m)m & \text{for even } N \text{ and } m \\ (N-m)m + d + 1 & \text{for even } N \text{ and odd } m \\ (N-m)m & \text{for odd } N \text{ and } m \end{array} \right\} \tag{81}$$

which exactly coincides (up to normalization) with the one-loop results obtained within the models of vector passive advection by the velocity field with a Gaussian statistics for both δ-correlated in time velocity field [24] and finite time correlations of the velocity field [33]. Therefore, all important conclusions about the persistence of large-scale anisotropy at small scales done in [24, 33] are also valid in our situation. Nevertheless, let us briefly analyze the problem within our model.

The persistence of the anisotropy in the inertial interval can be studied by the dimensionless ratio of the correlation functions of the field **b** (see, e.g., [24])

$$R_N = \frac{\left\langle b_r^{N-1}(t,\mathbf{x}) b_r(t,\mathbf{x}') \right\rangle}{\langle b_r(t,\mathbf{x}) b_r(t,\mathbf{x}')\rangle^{N/2}}. \tag{82}$$

Now, using the explicit expression (48), where dependence on the inner scale l and the outer scale L is considered, one comes to the following representation of R_N for even and odd values of N, namely,

$$R_{2n} \propto \left(\frac{r}{l}\right)^{-\gamma^*_{[2n-1,1]}} \left(\frac{r}{L}\right)^{\gamma^*_{[2n,0]} - n\gamma^*_{[2,0]}}, \tag{83}$$

$$R_{2n+1} \propto \left(\frac{r}{l}\right)^{-\gamma^*_{[2n,0]}} \left(\frac{r}{L}\right)^{\gamma^*_{[2n+1,1]} - (n+1/2)\gamma^*_{[2,0]}}, \tag{84}$$

where $\gamma^*_{[N,p]}$ is given in (73). The above expressions can be estimated as functions of the Reynolds number Re by using the definition $Re = (L/l)^{4/3}$ and by replacement $r \to l$ [43]

$$R_{2n} \propto Re^{\varepsilon \frac{n(n-1)}{(d+2)}}, \tag{85}$$

$$R_{2n+1} \propto Re^{\varepsilon \frac{4n^2-d-2}{4(d+2)}}, \tag{86}$$

and after substitution of the real value $\varepsilon = 2$ one obtains the final result

$$R_{2n} \propto Re^{2 \frac{n(n-1)}{(d+2)}}, \tag{87}$$

$$R_{2n+1} \propto Re^{\frac{4n^2-d-2}{2(d+2)}}, \tag{88}$$

which is equivalent to the result obtained within the kinematic MHD Kazantzev–Kraichnan model at one-loop level [24] by the replacement of the Reynolds number Re with the so-called Péclet number Pe.

The leading contribution to the even correlation functions (72) is given by the isotropic shell; therefore the behavior of the even functions R_N (82) is equal to the isotropic model as it is evident from (85) and (87). On the other hand, the nontrivial persistence of anisotropy deep inside of the inertial-range is manifested by the non-vanishing of the odd order correlation functions which must be equal to zero by the symmetry arguments in the isotropic case. This leads to the non-vanishing of the odd ratios (86) and (88), respectively. Among the odd functions R_N the function R_3 has special behavior, namely, it decreases for $Re \to \infty$. For example, for $d = 3$ one has $R_3 \propto Re^{-\frac{1}{10}}$. At the same time, the odd functions R_n for $n \geqslant 5$ increase with Re [24], e.g., for $d = 3$ one obtains $R_5 \propto Re^{\frac{11}{10}}$.

However, it must be stressed that the same asymptotic behavior of the models with Gaussian spatial statistics of the velocity field [24, 29] and non-Gaussian statistics of the velocity field (present paper) is an artifact of the one-loop approximation. Thus, it is necessary to investigate, at least, two-loop approximation to obtain more complete picture about behaviors of the different models. Unfortunately, for the present, no two-loop calculations of passively advected vector field by the Gaussian or non-Gaussian velocity fields were done. This situation is diametrically different from the passive scalar advection, where two-loop corrections (as for the simplest Kraichnan model also three-loop corrections are calculated [26]) are completely analyzed by the field theoretic RG technique (see, e.g., [25, 29, 36]).

5. Conclusion

Using the field theoretic RG technique and the OPE we have investigated the influence of uniaxial large-scale anisotropy on the behavior of a passive vector (e.g., weak magnetic field) advected by the non-Gaussian solenoidal velocity field governed by the stochastic Navier–Stokes equation in the framework of the so-called kinematic MHD turbulence in one-loop approximation. The coordinates of the stable IR fixed point are found which drive the IR asymptotic scaling behavior within the inertial range with definite exponents in accordance with the famous second Kolmogorov hypothesis of the KO theory. It is an exact one-loop result, i.e., it has no higher-loop corrections.

Further, we have investigated the influence of large-scale anisotropy on the anomalous scaling of the single-time correlation functions of a passively advected vector field using the OPE technique. The corresponding leading composite operators in the OPE with the smallest (the most negative) critical dimensions are identified and studied in detail to find their anomalous dimensions. Their hierarchical ordering in amount of anisotropy is briefly discussed. Consequently, these anomalous dimensions are used for the construction of the critical dimensions of the corresponding single-time correlation functions $B_{N-m,N}$ and their explicit dependence on the order of the correlation functions, as well as on the parameter which characterizes the anisotropy shells is established. The final one-loop asymptotic scaling behavior of the correlation functions deep inside the inertial range in the presence of the large-scale anisotropy is shown in (80) and (81). It is given by the anomalous dimensions of the operators from the isotropy shell and of the operators that are close to the isotropy shell. The persistence of anisotropy within inertial range is demonstrated by the behavior of the odd dimensionless ratios of the correlation functions (analogs of the skewness and hyperskewness built by the dimensionless ratios of structure functions of a passive scalar advected by given statistics of the velocity field) and their asymptotic behavior is established as functions of the Reynolds number Re.

Up to normalization factors, the obtained results are the same as one-loop expressions calculated within the toy models of passive vector advection by the 'synthetic' turbulent flows (see, e.g., [24, 33]). On the other hand, the anomalous dimensions of the important composite operators of our model, namely, of the operators built solely by the magnetic field **b** without derivatives, are the same as the one-loop anomalous dimensions of the composite operators built solely by the gradients of a scalar field within the models of a passive scalar advection by a Gaussian velocity field with δ-correlations in time [25], with finite correlations in time [29], as well as in the case, where the advection is given by a non-Gaussian velocity field governed by the stochastic Navier–Stokes equation [36].

However, it is evident that such kind of nondependence of anomalous behavior on the structure of the turbulent flow, as well as on the internal tensor structure of the advected field is an artifact of the one-loop approximation. As was shown in [36] within the advection of a passive scalar field, the two-loop corrections to the anomalous exponents in the model with realistic non-Gaussian velocity field are different from the corresponding exponents obtained within models with Gaussian velocity fields [25, 30]. The same situation, of course, is expected within the advection of the passive vector field although we have no two-loop results in this case. On the other hand, there is another interesting question which waits for an answer, namely: Is there nontrivial dependence of the anomalous exponents of an advected field on its internal tensor structure? Within RG technique this question can be directly tested at the two-loop level results but they are absent at the moment. Thus, the question is still open.

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References

[1] Monin A S and Jaglom A M 1975 *Statistical Fluid Mechanics: Mechanics of Turbulence vol 2* (Cambridge: MIT Press)
[2] McComb W D 1990 *The Physics of Fluid Turbulence* (Oxford: Clarendon)
[3] Frisch U 1995 *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge: Cambridge University Press)
[4] Sreenivasan K R and Antonia R A 1997 *Ann. Rev. Fluid Mech.* **29** 435
[5] Falkovich G, Gawędzki K and Vergassola M 2001 *Rev. Mod. Phys.* **73** 913
[6] Antonia R A, Satyaprakash B R and Hussain A K F 1982 *J. Fluid Mech.* **119** 55
Alselmet F, Gagne Y, Hopfinger E and Antonia R A 1984 *J. Fluid Mech.* **140** 63
Meneveau C and Sreenivasan K R 1990 *Phys. Rev.* A **41** 2246
Borgas M S 1992 *Phys. Fluids* A **4** 2055
Kuznetsov V R and Sabel'nikov V A 1990 *Turbulence and Combustion* (New York: Hemisphere)
[7] Zinn-Justin J 1989 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)

[8] Vasil'ev A N 2004 *Quantum-Field Renormalization Group in the Theory of Critical Phenomena and Stochastic Dynamics* (Boca Raton, FL: Chapman and Hall/CRC)
[9] De Dominicis C and Martin P C 1979 *Phys. Rev.* A **19** 419
[10] Adzhemyan L Ts, Vasil'ev A N and Pis'mak Yu M 1983 *Theor. Math. Phys.* **57** 1131
[11] Adzhemyan L Ts, Antonov N V and Vasil'ev A N 1996 *Phys. Usp.* **39** 1257
[12] Adzhemyan L Ts, Antonov N V and Vasil'ev A N 1999 *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (London: Gordon and Breach)
[13] Antonia R A, Hopfinger E J, Gagne Y and Anselmet F 1984 *Phys. Rev.* A **30** 2704
[14] Sreenivasan K R 1991 *Proc. R. Soc. London* A **434** 165
[15] Holzer M and Siggia E D 1994 *Phys. Fluids* **6** 1820
[16] Pumir A 1994 *Phys. Fluids* **6** 2118
[17] Tong C and Warhaft Z 1994 *Phys. Fluids* **6** 2165
[18] Elperin T, Kleeorin N and Rogachevskii I 1995 *Phys. Rev.* E **52** 2617
Elperin T, Kleeorin N and Rogachevskii I 1996 *Phys. Rev. Lett.* **76** 224
Elperin T, Kleeorin N and Rogachevskii I 1996 *Phys. Rev.* E **53** 3431
[19] Kraichnan R H 1968 *Phys. Fluids* **11** 945
[20] Gawedzki K and Kupiainen A 1995 *Phys. Rev. Lett.* **75** 3834
Bernard D, Gawedzki K and Kupiainen A 1996 *Phys. Rev.* E **54** 2564
Chertkov M, Falkovich G, Kolokolov I and Lebedev V 1995 *Phys. Rev.* E **52** 4924
Chertkov M and Falkovich G 1996 *Phys. Rev. Lett.* **76** 2706
[21] Kasantzev A P 1967 *Zh. Eksp. Teor. Fiz.* **53** 1806
[22] Vergassola M 1996 *Phys. Rev.* E **53** R3021
[23] Rogachevskii I and Kleeorin N 1997 *Phys. Rev.* E **56** 417
[24] Antonov N V, Lanotte A and Mazzino A 2000 *Phys. Rev.* E **61** 6586
[25] Adzhemyan L Ts, Antonov N V and Vasil'ev A N 1998 *Phys. Rev.* E **58** 1823
Adzhemyan L Ts, Antonov N V and Vasil'ev A N 1999 *Theor. Math. Phys.* **120** 1074
[26] Adzhemyan L Ts, Antonov N V, Barinov V A, Kabrits Yu S and Vasil'ev A N 2001 *Phys. Rev.* E **64** 056306
Adzhemyan L Ts, Antonov N V, Barinov V A, Kabrits Yu S and Vasil'ev A N 2001 *Phys. Rev.* E **63** 025303
[27] Adzhemyan L Ts, Antonov N V, Hnatich M and Novikov S V 2000 *Phys. Rev.* E **63** 016309
[28] Adzhemyan L Ts and Antonov N V 1998 *Phys. Rev.* E **58** 7381
Antonov N V and Honkonen J 2001 *Phys. Rev.* E **63** 036302
[29] Antonov N V 1999 *Phys. Rev.* E **60** 6691
[30] Antonov N V 2000 *Physica* D **144** 370
Antonov N V 2000 *Zap. Nauchn. Semin. POMI* **269** 79
Adzhemyan L Ts, Antonov N V and Honkonen J 2002 *Phys. Rev.* E **66** 036313
Hnatič M, Jurčišinová E, Jurčišin M and Repašan M 2006 *J. Phys. A: Math. Gen.* **39** 8007
Jurčišinová E and Jurčišin M 2008 *Phys. Rev.* E **77** 016306
[31] Chkhetiani O G, Hnatich M, Jurčišinová E, Jurčišin M, Mazzino A and Repašan M 2006 *Czech. J. Phys.* **56** 827
Chkhetiani O G, Hnatich M, Jurčišinová E, Jurčišin M, Mazzino A and Repašan M 2006 *J. Phys. A: Math. Gen.* **39** 7913
Chkhetiani O G, Hnatich M, Jurčišinová E, Jurčišin M, Mazzino A and Repašan M 2006 *Phys. Rev.* E **74** 036310
[32] Antonov N V, Honkonen J, Mazzino A and Muratore-Ginanneschi P 2000 *Phys. Rev.* E **62** R5891
Adzhemyan L Ts, Antonov N V and Runov A V 2001 *Phys. Rev.* E **64** 046310
Hnatich M, Honkonen J, Jurcisin M, Mazzino A and Sprinc S 2005 *Phys. Rev.* E **71** 066312
Hnatic M, Jurcisin M, Mazzino A and Sprinc S 2002 *Acta Phys. Slov.* **52** 559
Novikov S V 2006 *J. Phys. A: Math. Gen.* **39** 8133
[33] Antonov N V, Hnatich M, Honkonen J and Jurcisin M 2003 *Phys. Rev.* E **68** 046306
[34] Antonov N V 2006 *J. Phys. A: Math. Gen.* **39** 7825
[35] Kraichnan R H 1964 *Phys. Fluids* **7** 1723
Kraichnan R H 1965 *Phys. Fluids* **8** 575
Chen S and Kraichnan R H 1989 *Phys. Fluids* A **1** 2019
L'vov V S 1991 *Phys. Rep.* **207** 1
Chaves M, Gawędzki K, Horvai P, Kupiainen A and Vergassola M 2003 *J. Stat. Phys.* **113** 643
[36] Adzhemyan L Ts, Antonov N V, Honkonen J and Kim T L 2005 *Phys. Rev.* E **71** 016303
[37] Einaudi G, Velli M, Politano H and Pouquet A 1996 *Astrophys. J.* **457** L113
[38] Adzhemyan L Ts, Antonov N V, Kompaniets M V and Vasil'ev A N 2003 *Int. J. Mod. Phys.* B **17** 2137
[39] Adzhemyan L Ts, Antonov N V and Vasil'ev A N 1989 *Zh. Eksp. Teor. Fiz.* **95** 1272

[40] Biferale L, Lanotte A S and Toschi F 2004 *Phys. Rev. Lett.* **92** 094503
Biferale L, Cencini M, Lanotte A S, Sbragaglie M and Toschi F 2004 *New. J. Phys.* **6** 37
[41] Martin P C, Siggia E D and Rose H A 1973 *Phys. Rev.* A **8** 423
De Dominicis C 1976 *J. Phys. (Paris), Colloq.* **37** C1–247
Janssen H K 1976 *Z. Phys.* B **23** 377
Bausch R, Janssen H K and Wagner H 1976 *Z. Phys.* B **24** 113
[42] Adzhemyan L Ts, Vasil'ev A N and Hnatich M 1984 *Theor. Math. Phys.* **58** 47
[43] Pumir A 1996 *Europhys. Lett.* **34** 25